STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM
AFDELING STATISTIEK

SP 101

How to survive a fixed number of fair bets.

W. Molenaar and E. A. van der Velde

(S 374)

Reprinted from The Annals of Mathematical Statistics
Vol. 38, No. 4, August, 1967
Printed in U.S.A.

HOW TO SURVIVE A FIXED NUMBER OF FAIR BETS[1]

By W. Molenaar and E. A. van der Velde

Mathematisch Centrum, Amsterdam

Suppose a gambler with initial capital b_0 wants to maximize his probability of still having a positive capital after n_0 successive independent bets, under two conditions: (a) the minimal stake is one dollar; (b) bets are fair and their probbility of success is at most $\frac{1}{2}$.

A bet is determined by the stake c and the odds k: the gambler wins $kc - c$ with probability $1/k$ and loses c otherwise. If b_{m-1} denotes the gambler's capital after $m - 1$ bets, he must choose for the mth bet c_m $(1 \leqq c_m \leqq b_{m-1})$ and k_m $(k_m \geqq 2)$. For simplicity of presentation we make the inessential restriction that all b_m, c_m and k_m are integers. In a fair roulette (without zero) k can only be a divisor of 36. A bet $c = 1$, $k = 2$ is called conservative.

A situation is a pair (n, b) where b is the capital and n the number of bets to go. A strategy for (n_0, b_0) is a rule prescribing which bet should be made in the initial situation (n_0, b_0) and in each situation which may evolve from it. Under the stated conditions there exists for each (n_0, b_0) a (possibly non-unique) optimal strategy which leads to a (unique) maximal probability of survival (pos) denoted by $p(n_0, b_0)$. The independence of bets implies that for $n > 1$ and $b \geqq 1$

$$p(n, b) = \max_{c,k} \{(1/k)p(n - 1, b + kc - c) + (1 - 1/k)p(n - 1, b - c)\}.$$

Theorem 1. *The* pos $q(n, b)$ *for the conservative strategy* (i.e. $c = 1$ *and* $k = 2$ *in each situation*) *is for every* $n \geqq 1$ *a concave function of* b.

Proof. The theorem holds for $n = 1$ as $q(1, 0) = 0$, $q(1, 1) = \frac{1}{2}$ and $q(1, b) = 1$ for $b \geqq 2$. We proceed by induction. The definition of q implies that

$$q(n - 1, \beta) \geqq q(n, \beta) \tag{1}$$

and

$$q(n, b) = \tfrac{1}{2}q(n - 1, b + 1) + \tfrac{1}{2}q(n - 1, b - 1). \tag{2}$$

Substituting (1) with $\beta = b \pm 1$ into (2) we obtain $q(n, \lambda\beta_1 + (1 - \lambda)\beta_2) \geqq \lambda q(n, \beta_1) + (1 - \lambda)q(n, \beta_2)$, first for $\lambda = \frac{1}{2}$ and then by well known arguments for all $\lambda \in (0, 1)$ and all β_1, β_2 such that both sides of the inequality are defined.

Theorem 2. *The conservative strategy is optimal for all* n_0 *and* b_0.

Received 5 December 1966.

[1] Report S 374 (Sp 101), Statistische Afdeling, Mathematisch Centrum, Amsterdam.

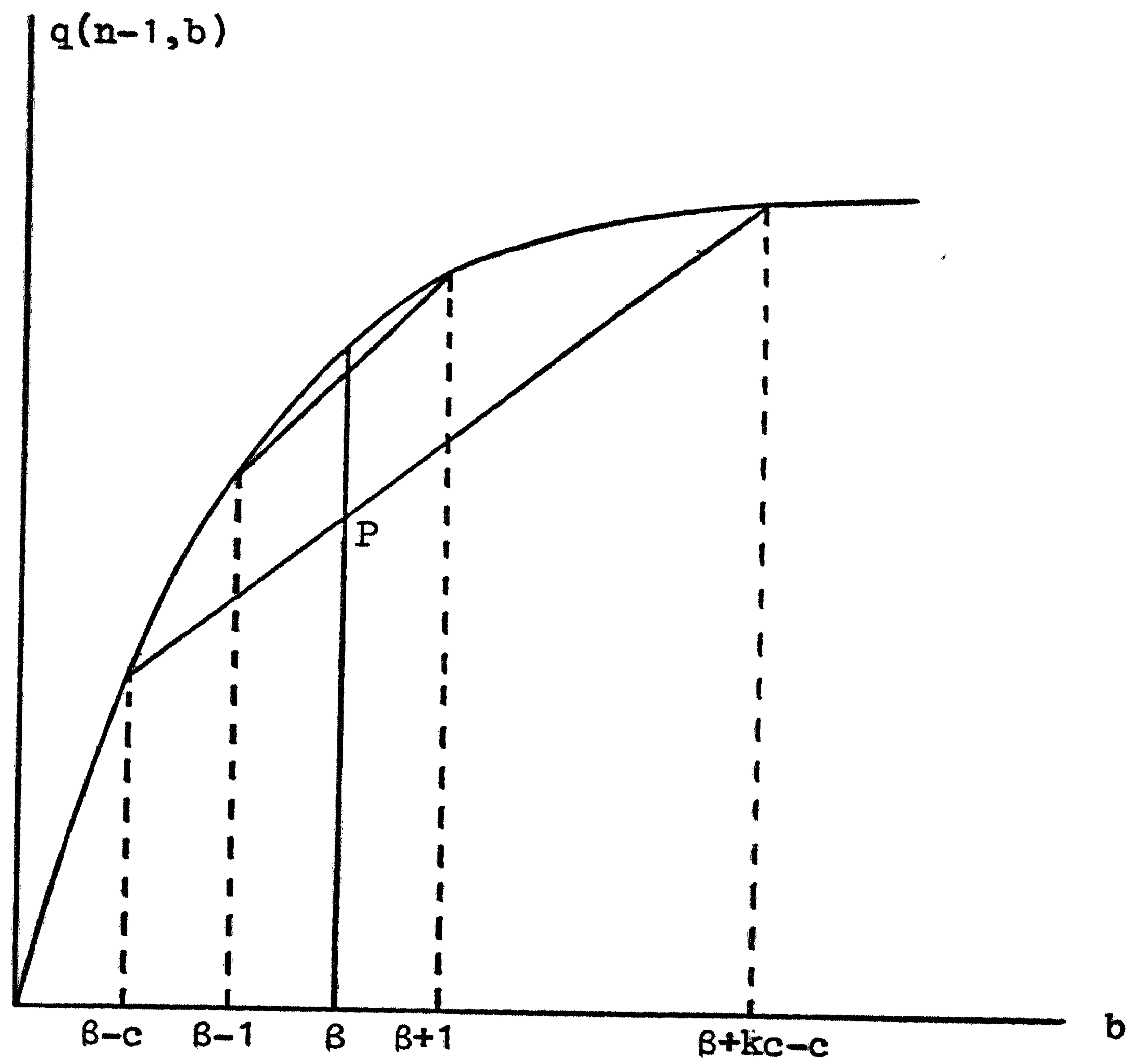

FIG. 1

PROOF. This is trivial for $n_0 = 1$. Suppose it holds for $n_0 = n - 1$. The pos from (n, β) for the bet (c, k) followed by $(n - 1)$ conservative bets is represented by the ordinate of the point of intersection P of the vertical in β and the chord connecting the points on the graph of $q(n - 1, \cdot)$ with abscissae $\beta - c$ and $\beta + kc - c$ (see Figure 1). As the function is concave, the choice $c = 1$, $k = 2$ is seen to be optimal under our conditions $c \geqq 1$, $k \geqq 2$.

REMARK 1. Very similar and somewhat more general results were obtained independently by Freedman [2].

REMARK 2. $q(n, b)$ is determined recursively from (2) and the boundary conditions $q(n, 0) = 0$ for all n, $q(0, b) = 1$ for all $b \geqq 1$. No closed expression for q seems to exist, but we have

$$q(n, b) = \sum_{j=n+1}^{\infty} \lambda_j^{(b)}$$

where $\lambda_j^{(b)}$ are the well-known first passage probabilities for the symmetric random walk given in [1]; p. 254–256.

REMARK 3. Suppose bets are unfair, in the sense that there is a fixed $\alpha < 1$

such that the gambler gains $kc - c$ with probability α/k, and loses c otherwise. It then turns out that bold bets become attractive for small α. For $n_0 = 3$, $b_0 = 1$ the conservative strategy is only optimal for $\alpha > 2 - 2/3^{\frac{1}{2}} \approx .84$. For $n_0 = 13$, $b_0 = 1$ an initial bet $c_1 = 1$, $k_1 = 3$ must be made even for an ordinary roulette with one zero ($\alpha = 36/37$).

REFERENCES

[1] FELLER, W. (1957). *An Introduction to Probability Theory and Its Applications* (2nd edition). **1** Wiley, New York.

[2] FREEDMAN, D. (1967). Timid play is optimal. *Ann. Math. Statist.* **38** 1281–1283.